14.2

1). $$\sum_{n\geq 2}\frac{1}{n^3-n} = \sum_{n\geq 2}\frac{1}{n(n^2-1)}$$

$$= \sum_{n\geq 2}\frac{1}{n(n-1)(n+1)}$$

$$= \sum_{n\geq 2}\left(\frac{1}{n-1}-\frac{1}{n}\right)\frac{1}{(n+1)}$$

$$= \sum_{n\geq 2}\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)-\frac{1}{n}+\frac{1}{n+1}$$

$$= \sum_{n\geq 2}\frac{1}{2}\left[\left(\frac{1}{n-1}-\frac{1}{n}\right)-\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$$

2). $$\sum_{n\geq 1}^{K}\ln\left(\frac{n^2+3n+2}{n^2+3n}\right) = \sum_{n\geq 1}^{K}\ln\left(\frac{(n+1)(n+2)}{n(n+3)}\right)$$

$$= \sum_{n\geq 1}^{K}\ln\left(\frac{n+1}{n}\right)-\ln\left(\frac{n+3}{n+2}\right)$$

$$= \ln(k+1)-\ln\frac{k+3}{3}$$

$$= \ln 3+\ln\frac{k+1}{k+3}$$

$$\Rightarrow \sum_{n\geq 1}\ln\left(\frac{n^2+3n+2}{n^2+3n}\right) = \lim_{k\to+\infty}\sum_{n\geq 1}^{K}\ln\left(\frac{n^2+3n+2}{n^2+3n}\right) = \ln 3.$$

3). $$\sum_{n\geq 0}\left((3+(-1)^n)\right)^n$$

When n is even, $(3+(-1)^n)^{-n} = 4^{-n}$

When n is odd, $(3+(-1)^n)^{-n} = 2^{-n}$.

$\Rightarrow \sum_{n\geq 0}\left((3+(-1)^n\right)^n \leq \sum_{n\geq 0}2^{-n}$ is convergent.

$$\Rightarrow \sum_{n\geq 0}(3+(-1)^n)^n = \sum_{n=2K}(3+(-1)^n)^{-n} + \sum_{n=2k+1}(3+(-1)^n)^n$$

$$= \sum_{K=0}^{\infty}4^{-2K}+\sum_{K=0}^{\infty}2^{-(2k+1)}$$

$$= \sum_{K=0}^{\infty}16^{-K}+\frac{1}{2}\sum_{K=0}^{\infty}4^{-K}$$

$$= \frac{1}{1-\frac{1}{16}}+\frac{1}{2}\frac{1}{1-\frac{1}{4}} = \frac{16}{15}+\frac{1}{2}\cdot\frac{4}{3} = \frac{26}{15}$$

14.4. $\forall p\in\mathbb{N}, n\in\mathbb{N}$, set $u_{p,n} = \frac{1}{\binom{n+p}{n}}$

Thus $$u_{p,n} = \frac{n\cdot(n-1)\cdots 1}{(p+n)(p+n-1)\cdots(p+1)}$$

$$= \frac{(p+n-p)\cdot(n-1)\cdots 1}{(p+n)(p+n-1)\cdots(p+1)}$$

$$= \frac{1}{\binom{n-1+p}{n-1}} - \frac{p}{p+1}\cdot\frac{1}{\binom{n+p}{n-1}} = u_{p,n-1} - \frac{p}{p+1}u_{p+1,n-1}.$$

$\Rightarrow u_{p+1,n-1} = \frac{p+1}{p}(u_{p,n-1} - u_{p,n})$

When $p=0$, $u_{0,n}=1$, $\sum_{n\in\mathbb{N}} u_{0,n}$ is divergent.

When $p=1$, $u_{1,n} = \frac{1}{n+1}$, $\sum_{n\in\mathbb{N}} u_{1,n}$ is divergent.

When $p\geq 2$, $u_{p,n} = \frac{p}{p-1}(u_{p-1,n} - u_{p-1,n+1})$, thus

$$\sum_{n\geq 0}^{\infty} u_{p,n}$$

$$= \frac{p}{p-1}\left(u_{p-1,0} - \lim_{n\to\infty} u_{p-1,n}\right)$$

$$= \frac{p}{p-1} - \frac{p}{p-1}\lim_{n\to\infty}\frac{n\cdot(n-1)\cdots 1}{(p+n-1)(p+n-2)\cdots p} \quad \left(n\geq p,\ \frac{n\cdot(n-1)\cdots 1}{(p+n-1)(p+n-2)\cdots p} = \frac{(p-1)!}{(p+n-1)\cdots(n+1)} \leq \frac{(p-1)!}{p+n-1} \to 0 \text{ as } n\to\infty \right)$$

$$= \frac{p}{p-1}$$

In summary, $$\sum_{n\geq 0} u_{p,n} = \begin{cases} +\infty & p=0,1 \\ \frac{p}{p-1} & p\geq 2 \end{cases}$$

$\square$

14.6. $u_n = \sqrt{n} + a\sqrt{n+1} + b\sqrt{n+2}$

解:

① $(a,b) = (-2,1)$.

$$u_n = \sqrt{n} - 2\sqrt{n+1} + \sqrt{n+2}$$

$$= (\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n})$$

$$= \frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\sum_{n\geq 0} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+2}+\sqrt{n+1}} - 1 = -1$$

② $(a,b) \neq (-2,1)$,

Consider $v_n = u_n - (\sqrt{n} - 2\sqrt{n+1} + \sqrt{n+2})$

$= (a+2)\sqrt{n+1} + (b-1)\sqrt{n+2}$

Then $\sum_{n\geq 0} u_n = \sum_{n\geq 0} v_n - 1$

We rewrite

$$\frac{v_n}{a+2} = \sqrt{n+1} - \sqrt{n+2} + A\sqrt{n+2}$$

where $A = \frac{b-1}{a+2} + 1 \neq 1$.

If $A = 0$, $\frac{v_n}{a+2} = \sqrt{n+1} - \sqrt{n+2} \leq \frac{-2}{\sqrt{n+2}} \Rightarrow \frac{1}{a+2}\sum_{n\geq 0} u_n = \frac{1}{a+2}\left(\sum_{n\geq 0} v_n - 1\right) = -\infty$

If $A > 0$, $\frac{v_n}{a+2} = \sqrt{n+1} - \sqrt{n+2} + A\sqrt{n+2}$

$= A\sqrt{n+2} - \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

$\geq A\sqrt{n+2} - \frac{2}{\sqrt{n+1}} = \frac{1}{\sqrt{n+2}}(A(n+2) - 2)$

$\geq \frac{2}{\sqrt{n+2}}$ for $n \geq \frac{2}{A} - 2$

$\Rightarrow \frac{1}{a+2}\sum_{n\geq 0} u_n = \frac{1}{a+2}\left(\sum_{n\geq 0} v_n - 1\right) = +\infty$.

If $A < 0$, $\frac{v_n}{a+2} \leq A\sqrt{n+2} \Rightarrow \frac{1}{a+2}\sum_{n\geq 0} u_n = \frac{1}{a+2}\left(\sum_{n\geq 0} v_n - 1\right) = -\infty$

In summary, $\sum_{n\geq 0} u_n = \begin{cases} -1 & (a,b) = (-2,1). \\ \infty & (a,b) \neq (-2,1) \end{cases}$

14.8.

Pf: $0 \leq \sum_{n\geq 0} \max\{u_n, v_n\} \leq \sum u_n + \sum v_n \Rightarrow \sum_{n\geq 0} \max\{u_n, v_n\}$ is convergent.

$0 \leq \sum_{n\geq 0} \sqrt{u_n v_n} \leq \sum_{n\geq 0} \frac{u_n + v_n}{2} = \frac{1}{2}\sum_{n\geq 0} u_n + \frac{1}{2}\sum_{n\geq 0} v_n \Rightarrow \sum_{n\geq 0} \sqrt{u_n v_n}$ is convergent.

14.10. $n \in \mathbb{N}^*$, $u_n = \begin{cases} -\frac{4}{n}, & 5 \mid n. \\ \frac{1}{n}, & 5 \nmid n. \end{cases}$

1). $\sum_{5(n-1)+1}^{5n} = \frac{1}{5n-4}+\frac{1}{5n-3}+\frac{1}{5n-2}+\frac{1}{5n-1}-\frac{4}{5n}$

$= \frac{1}{5n-4}+\frac{1}{5n-3}+\frac{1}{5n-2}+\frac{1}{5n-1}+\frac{1}{5n}-\frac{5}{5n}$ (桂昊博)

$\sum_{k=1}^{5n} u_k = \sum_{k=1}^{5n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=n+1}^{5n}\frac{1}{k}$

$\ln\frac{5n+1}{n+1} = \sum_{k=n+1}^{5n}\int_k^{k+1}\frac{1}{x}dx < \sum_{k=n+1}^{5n}\frac{1}{k} < \sum_{k=n+1}^{5n}\int_{k-1}^{k}\frac{1}{x}dx = \int_n^{5n}\frac{1}{x}dx = \ln 5$

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{5n} u_k = \ln 5$

2). $\forall n\in\mathbb{N}^*$, $n\in[5([\frac{n}{5}]), 5([\frac{n}{5}]+1))$

$\Rightarrow \ln\frac{5[\frac{n}{5}]+1}{[\frac{n}{5}]+1} < \sum_{k=1}^{5[\frac{n}{5}]} u_k < \sum_{k=1}^{n} u_k < \sum_{k=1}^{5[\frac{n}{5}]} u_k + \frac{4}{5[\frac{n}{5}]+1} < \ln 5 + \frac{4}{5[\frac{n}{5}]+1}$

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n} u_k = \ln 5$

$\square$.

14.12. $x\in]0,\frac{\pi}{2}[$, $n\in\mathbb{N}$, $u_n=\ln(\cos\frac{x}{2^n})$

Pf:

$\ln(\cos\frac{x}{2^n})+\ln(\sin\frac{x}{2^n}) = \ln(\sin\frac{x}{2^n}\cdot\cos\frac{x}{2^n}) = \ln(\frac{1}{2}\cdot\sin\frac{x}{2^{n-1}})$

$\Rightarrow \ln(\sin\frac{x}{2^n}) + \sum_{k=0}^{n}\ln(\cos\frac{x}{2^k})$

$= \ln\frac{1}{2}+\ln(\sin\frac{x}{2^{n-1}}) + \sum_{k=1}^{n-1}\ln(\cos\frac{x}{2^k})$

$= n\cdot\ln\frac{1}{2}+\ln(\sin x) = \ln\frac{1}{2^n}+\ln(\sin x)$

$\Rightarrow \sum_{k=0}^{n}\ln(\cos\frac{x}{2^k}) = \ln(\sin x)+\ln\frac{\frac{1}{2^n}}{\sin\frac{x}{2^n}}$

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\ln(\cos\frac{x}{2^k}) = \ln\frac{\sin x}{x}$

14.14. $\sum_{n\geq 1} \frac{\ln n}{n}$

Pf: $\forall x \geq 1,\ \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \Rightarrow \left(\frac{\ln x}{x}\right)' < 0,\ \forall x \geq 3$

$\Rightarrow \forall n \geq 3,$

$$\int_n^{n+1} \frac{\ln x}{x} dx < \frac{\ln n}{n} < \int_{n-1}^{n} \frac{\ln x}{x} dx$$

$$\Rightarrow \quad \int_3^{n+1} \frac{\ln x}{x} dx < \sum_{k\geq 3}^{n} \frac{\ln n}{n} < \int_2^{n} \frac{\ln x}{x} dx$$

$$\Rightarrow \frac{1}{2}(\ln(n+1))^2 - \frac{1}{2}(\ln 3)^2 < \sum_{k\geq 3}^{n} \frac{\ln n}{n} < \frac{1}{2}(\ln n)^2 - \frac{1}{2}(\ln 2)^2$$

$$\Rightarrow \sum_{k=1}^{n} \frac{\ln n}{n} \sim \frac{1}{2}(\ln(n+1))^2$$

□

14.16. $\alpha \in \mathbb{R},\ u_n = \left(\cos\frac{1}{n}\right)^{n^\alpha}$

Pf:

$$\left(\cos\frac{1}{n}\right)^{n^\alpha} = \left(1 - 2\sin^2\frac{1}{2n}\right)^{n^\alpha}$$

$$= \left(1 - 2\sin^2\frac{1}{2n}\right)^{-\frac{1}{2\sin^2\frac{1}{2n}} \cdot \left(-2\left(\sin^2\frac{1}{2n}\right)\cdot n^\alpha\right)}$$

$$= e^{-2\left(\sin^2\frac{1}{2n}\right)\cdot n^\alpha \cdot \frac{\ln\left(1-2\sin^2\frac{1}{2n}\right)}{-2\sin^2\frac{1}{2n}}}$$

Where $-2\left(\sin^2\frac{1}{2n}\right)\cdot n^\alpha \cdot \frac{\ln\left(1-2\sin^2\frac{1}{2n}\right)}{-2\sin^2\frac{1}{2n}} = -\frac{n^{\alpha-2}}{2}\left(1+o\left(\frac{1}{n}\right)\right)$

$\Rightarrow \exists\, n_0 \in \mathbb{N}$ sufficiently large s.t. $\forall n \geq n_0$

$$\left(\frac{1}{e^{\frac{1}{4}}}\right)^{n^{\alpha-2}} < \left(\cos\frac{1}{n}\right)^{n^\alpha} = \left(\frac{1}{\sqrt{e}}\right)^{n^{\alpha-2}\left(1-\frac{1}{48n^2}+o\left(\frac{1}{n^2}\right)\right)} < \left(\frac{1}{\sqrt{e}}\right)^{n^{\alpha-2}}$$

$\Rightarrow$ If $\alpha < 2$, $\lim_{n\to\infty}\left(\cos\frac{1}{n}\right)^{n^\alpha} = 1 \Rightarrow \sum\left(\cos\frac{1}{n}\right)^{n^\alpha} = +\infty$

If $\alpha = 2$, $\lim_{n\to\infty}\left(\cos\frac{1}{n}\right)^{n^\alpha} \geq \frac{1}{e^{\frac{1}{4}}} \Rightarrow \sum\left(\cos\frac{1}{n}\right)^{n^\alpha} = +\infty$

If $\alpha < 2$, $\forall n \geq n_0$ $\left(\frac{1}{\sqrt{e}}\right)^{n^{\alpha-2}} < \int_{n^{\alpha-2}}^{(n+1)^{\alpha-2}} \left(\frac{1}{\sqrt{e}}\right)^x dx$

$$\sum_{n\geq n_0}^{n}\left(\frac{1}{\sqrt{e}}\right)^{n^{\alpha-2}} < \int_{n_0^{\alpha-2}}^{(n+1)^{\alpha-2}}\left(\frac{1}{\sqrt{e}}\right)^x dx$$

$$= \frac{1}{\ln\left(\frac{1}{\sqrt{e}}\right)}\left(\frac{1}{\sqrt{e}}\right)^x \Big|_{n_0^{\alpha-2}}^{(n+1)^{\alpha-2}}$$

$$< \frac{1}{2}\left(\frac{1}{\sqrt{e}}\right)^{n_0^{\alpha-2}}$$

In summary, $\sum\left(\cos\frac{1}{n}\right)^{n^\alpha} = \begin{cases} <+\infty & \alpha<2 \\ +\infty & \alpha\geq 2. \end{cases}$

□

14.18.

1). pf: $S_{2n}-S_{2(n-1)} = V_{2n}-V_{2n-1} < 0$

$S_{2n+1}-S_{2(n-1)+1} = -V_{2n+1}+V_{2n} > 0$

On the other hand, $S_{2n+1}-S_{2n} = -V_{2n+1} < 0$

$\Rightarrow$

$$S_1 < S_3 < S_5 < \cdots < S_{2n+1} < S_{2n} < \cdots < S_4 < S_2 < S_0.$$

□

2). pf: $\forall n\in\mathbb{N},\ n\in[2[\frac{n}{2}], 2[\frac{n}{2}]-1)$

$\Rightarrow \sum_{k\geq 0}^{2[\frac{n}{2}]+1} u_k < \sum_{k\geq 0}^{n} u_k < \sum_{k\geq 0}^{2[\frac{n}{2}]} u_k$

□

3). pf: If $\alpha>0$, let $V_n=\frac{1}{n^\alpha}$, $u_n=(-1)^n\frac{1}{n^\alpha}$, we can apply (2) to get

$\sum_{n\geq 0}(-1)^n\frac{1}{n^\alpha}$ is convergent.

If $\alpha\leq 0$, $\lim_{n\to\infty}\frac{(-1)^n}{n^\alpha}\neq 0 \Rightarrow \sum_{n\geq 0}(-1)^n\frac{1}{n^\alpha}$ is divergent.

14.19.

Pf: $\forall n\in\mathbb{N}_+$, $e^x = \sum_{k=0}^{n}\frac{x^n}{n!}+\frac{e^y}{(n+1)!}x^{n+1}$, for some $y\in[0,x]$

$$\Rightarrow \quad \sum_{k=0}^{n} \frac{x^n}{n!} \leq e^x \leq \sum_{k=0}^{n} \frac{x^n}{n!} + \frac{e^x}{(n+1)!} x^{n+1}$$

$$\Rightarrow \quad e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\square$